# Technical Report 325

L. Kleinrock

## Detection of the Energy Peak of an Arbitrary Signal

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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# DETECTION OF THE ENERGY PEAK OF AN ARBITRARY SIGNAL

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*Group 28*

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# ABSTRACT

A new procedure is described for determining that frequency at which the power spectrum of a signal has its absolute peak. The salient feature of the procedure is that it does not explicitly involve the estimation of the power spectrum of the signal itself. Specifically, it is shown that the limit of the iterated normalized autocorrelation [see Eqs. (4) and (5)] of a function f(t) is a pure cosine wave whose frequency corresponds to the location of the maximum energy density in the spectrum of f(t).

Furthermore, if one is willing to accept the frequency of maximum energy to within a given finite spectral resolution, then the procedure terminates after a specified finite number of iterations. Results from a computer simulation of the procedure are also described.

The areas of application of this procedure are discussed. The results indicate that this method of detecting a signal (i.e., by the peak of its spectrum) merits further consideration.

It is important to note that the effects of noise have not been considered in this initial study; the results apply to the received signal only.

This technical documentary report is approved for distribution.

Franklin C. Hudson

Franklin C. Hudson, Deputy Chief
Air Force Lincoln Laboratory Office

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# DETECTION OF THE ENERGY PEAK OF AN ARBITRARY SIGNAL

## I. INTRODUCTION

In communications engineering, it is often useful to be able to extract, from an incoming signal, that frequency which contains a greater energy density than any other frequency. In this study, a new procedure is described for determining that frequency; the significant aspect of the procedure is that the power spectrum of the signal need not be calculated.

The fundamental theorem of Sec. III describes the mathematically interesting result in the case where we carry our procedure to the limit. The more useful theorem (Theorem 2, Sec. V), however, describes a realizable procedure for determining the frequency of maximum energy to within a finite spectral resolution.

We begin by defining the quantities basic to the procedure.

## II. DEFINITIONS

Consider that class of real functions or signals f(t) whose autocorrelation function g(t) has the following properties:

$$0 < g(0) < \infty \quad , \tag{1}$$

$$\int_{-\infty}^{\infty} |g(t)| \, dt < \infty \quad , \tag{2}$$

$$g(t) \text{ is continuous in } t \quad . \tag{3}$$

Since g(t) is an autocorrelation, it is therefore an even function of its argument. We now define a set of normalized autocorrelation functions $R_n(t)$ as

$$R_o(t) = \frac{g(t)}{g(0)} \tag{4}$$

and

$$R_n(t) = \frac{\int_{-\infty}^{\infty} R_{n-1}(\tau)\, R_{n-1}(t+\tau)\, d\tau}{\int_{-\infty}^{\infty} R_{n-1}^2(\tau)\, d\tau} \qquad n = 1, 2, 3, \ldots \quad . \tag{5}$$

We recognize that $R_1(t)$ is the normalized autocorrelation function of the normalized autocorrelation function $R_o(t)$, etc. We may thus consider $R_n(t)$ to be the $n^{th}$ iterated normalized autocorrelation function of f(t).

Consider the Fourier transform $S_n(\omega)$, defined by

$$S_n(\omega) = \int_{-\infty}^{\infty} R_n(t)\, e^{-j\omega t}\, dt \tag{6}$$

and its inverse

$$R_n(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_n(\omega)\, e^{j\omega t}\, d\omega \quad . \tag{7}$$

Note that $S_n(\omega)$ is proportional to the spectral density of $R_{n-1}(t)$; in particular, $S_o(\omega)$ is proportional to the spectral density† of f(t). We also observe that $R_n(t)$ and $S_n(\omega)$ are both real, even functions of their arguments.

## III. THE FUNDAMENTAL THEOREM

In Sec. I we stated that our procedure, consisting of operations limited strictly to the time domain, enables us to determine the peak of the spectrum of a function f(t). We will now state the theorem involved.

**Theorem 1.**

If $S_o(\omega)$ has a unique absolute maximum at $\omega = \Theta$, <u>viz</u>.,

$$S_o(\Theta) > S_o(\omega) \quad \text{for } \omega \neq \pm\Theta \quad , \tag{8}$$

then

$$\lim_{n\to\infty} R_n(t) = \cos \Theta t \tag{9}$$

and

$$\frac{S_n(\omega)}{S_n(\Theta)} = \left[\frac{S_o(\omega)}{S_o(\Theta)}\right]^{2^n} \quad . \tag{10}$$

**Proof.**

We first demonstrate that $S_n(\omega)$ exists for all finite n. A sufficient condition‡ for the existence of $S_n(\omega)$ is

$$\int_{-\infty}^{\infty} |R_n(t)|\, dt < \infty \quad . \tag{11}$$

We will establish this condition by an inductive proof on the hypothesis $H_n$ which states that $R_n(t)$ is integrable and continuous in t. Clearly, by Eqs. (1) through (4), we have $H_o$. We now assume $H_{n-1}$ and show that this implies $H_n$. From Eq. (5) we may form

$$\int_{-\infty}^{\infty} |R_n(t)|\, dt = \frac{\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} R_{n-1}(\tau)\, R_{n-1}(t+\tau)\, d\tau \right| dt}{\int_{-\infty}^{\infty} R_{n-1}^2(\tau)\, d\tau} \quad . \tag{12}$$

† Whether this is the energy density or power density depends upon the class that f(t) falls in. See Eqs. (25) through (28) and the discussion following Eq. (28).

‡ See Ref. 1, p. 103.

But

$$\left|\int_{-\infty}^{\infty} R_{n-1}(\tau)\, R_{n-1}(t+\tau)\, d\tau\right| \leqslant \int_{-\infty}^{\infty} |R_{n-1}(\tau)|\, |R_{n-1}(t+\tau)|\, d\tau \quad ,$$

thus

$$\int_{-\infty}^{\infty} |R_n(t)|\, dt \leqslant \frac{\left[\int_{-\infty}^{\infty} |R_{n-1}(\tau)|\, d\tau\right]^2}{\int_{-\infty}^{\infty} R_{n-1}^2(\tau)\, d\tau} \quad . \tag{13}$$

However, by assumption, $\int_{-\infty}^{\infty} |R_{n-1}(\tau)|\, d\tau < \infty$. Furthermore, since $R_{n-1}(t)$ is assumed to be continuous and $R_{n-1}(0) = 1$ [from Eq. (5)], then $\int_{-\infty}^{\infty} R_{n-1}^2(\tau)\, d\tau > 0$. This establishes Eq. (11). Now, since for normalized autocorrelation functions†

$$R_{n-1}(t) \leqslant R_{n-1}(0) = 1 \quad , \tag{14}$$

we may write

$$R_{n-1}(\tau)\, R_{n-1}(t+\tau) \leqslant R_{n-1}(\tau) \quad . \tag{15}$$

Therefore, we may bound from above the integrand of the numerator in Eq. (5) by a function which is itself integrable and continuous (by assumption of $H_{n-1}$). Thus, $R_n(t)$ is continuous.‡ This proves $H_n$ and completes the inductive proof, thus establishing the existence of $S_n(\omega)$ for all fixed n.

We now use the fact that the Fourier transform of the autocorrelation of a real, even function is equal to the square of the Fourier transform of the function itself. In our case, we have already established the existence of the Fourier transform of the function $R_n(t)$, namely, $S_n(\omega)$. Thus, since the numerator of Eq. (5) represents the autocorrelation of the function $R_{n-1}(t)$, we transform both sides of Eq. (5) and make use of the product relation above to obtain

$$S_n(\omega) = \frac{S_{n-1}^2(\omega)}{\int_{-\infty}^{\infty} R_{n-1}^2(\tau)\, d\tau} \qquad n = 1, 2, 3, \ldots \quad . \tag{16}$$

We now concentrate our attention on the frequency $\Theta$. Forming the ratio $S_n(\omega)/S_n(\Theta)$, we obtain from Eq. (16)

$$\frac{S_n(\omega)}{S_n(\Theta)} = \left[\frac{S_{n-1}(\omega)}{S_{n-1}(\Theta)}\right]^2 \quad . \tag{17}$$

It is then clear that

$$\frac{S_n(\omega)}{S_n(\Theta)} = \left[\frac{S_o(\omega)}{S_o(\Theta)}\right]^{2^n} \quad ,$$

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† See Ref. 1, p. 61.

‡ See Ref. 2, p. 67.

which establishes Eq. (10). We develop some alternate expressions for $S_n(\omega)$ as follows. Let

$$\alpha_{n-1}^2 = \frac{1}{\int_{-\infty}^{\infty} R_{n-1}^2(\tau)\, d\tau} \quad .$$

Then Eq. (16) may be written as

$$S_n(\omega) = K_n S_o^{2^n}(\omega) \quad , \tag{18}$$

where

$$K_n = \alpha_{n-1}^2 \, \alpha_{n-2}^{2^2}, \ldots, \alpha_o^{2^n} \quad .$$

But, by Eqs. (5) and (7),

$$R_n(0) = 1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_n(\omega)\, d\omega \quad .$$

Integrating Eq. (18) and applying the above relation, we see that

$$K_n = \frac{2\pi}{\int_{-\infty}^{\infty} S_o^{2^n}(\omega)\, d\omega} \quad .$$

Thus, we obtain, as alternate forms for $S_n(\omega)$,

$$S_n(\omega) = 2\pi \frac{S_o^{2^n}(\omega)}{\int_{-\infty}^{\infty} S_o^{2^n}(\sigma)\, d\sigma} \quad , \tag{19}$$

and

$$S_n(\omega) = 2\pi \frac{[S_o(\omega)/S_o(\Theta)]^{2^n}}{\int_{-\infty}^{\infty} [S_o(\sigma)/S_o(\Theta)]^{2^n}\, d\sigma} \quad .$$

We now proceed to establish Eq. (9). By assumption

$$\frac{S_o(\omega)}{S_o(\Theta)} < 1 \qquad \text{for all } \omega \neq \pm\Theta \quad . \tag{20}$$

Thus, by Eqs. (10) and (20), we see that

$$\lim_{n\to\infty} \frac{S_n(\omega)}{S_n(\Theta)} = \begin{cases} 0 & \omega \neq \pm\Theta \\ 1 & \omega = \pm\Theta \end{cases} \quad . \tag{21}$$

Furthermore, it is clear from Eqs. (7) and (14) that for all fixed n,

$$\int_{-\infty}^{\infty} S_n(\omega)\, d\omega = 2\pi R_n(0) = 2\pi \quad . \tag{22}$$

Since the integral of $S_n(\omega)$ is constant and since $S_n(\omega)$ is vanishingly small compared to $S_n(\Theta)$ at all $\omega \neq \pm\Theta$, we recognize that the limit at $n \to \infty$ must be†

$$\lim_{n\to\infty} S_n(\omega) = \pi u_o(\omega - \Theta) + \pi u_o(\omega + \Theta) \quad . \tag{23}$$

The transform of this limit function (by Eq. 7) is

$$\lim_{n\to\infty} R_n(t) = \cos \Theta t \quad ,$$

which establishes Eq. (9) and completes the proof of Theorem 1.

**Corollary.**

If $S_o(\omega)$ has a finite number of equal absolute maxima at the frequencies $\Theta_k$ $(k = 1, 2, \ldots, K)$, then

$$\lim_{n\to\infty} R_n(t) = \frac{1}{K} \sum_{k=1}^{K} \cos \Theta_k t \quad ,$$

and

$$\frac{S_n(\omega)}{S_n(\Theta_k)} = \left[\frac{S_o(\omega)}{S_o(\Theta_k)}\right]^{2^n} \quad .$$

**Proof.**

The proof here is identical with that of Theorem 1 through Eq. (20) (where $\Theta_k$ is substituted for $\Theta$). We now recognize that the limit of $S_n(\omega)$ must be

$$\lim_{n\to\infty} S_n(\omega) = \frac{\pi}{K} \sum_{k=1}^{K} [u_o(\omega - \Theta_k) + u_o(\omega + \Theta_k)] \quad . \tag{24}$$

The transform of this limit is clearly

$$\lim_{n\to\infty} R_n(t) = \frac{1}{K} \sum_{k=1}^{K} \cos \Theta_k t \quad ,$$

which completes the proof of the corollary.

Let us now consider those functions f(t) whose autocorrelation function g(t) has the properties listed in Eqs. (1) to (3). We define functions f(t) to be of <u>finite energy</u> if

$$0 < \int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty \quad , \tag{25}$$

in which case

$$g(t) = \int_{-\infty}^{\infty} f(\tau)\, f(t + \tau)\, d\tau \quad . \tag{26}$$

† Where $u_o(x)$ is a unit impulse occurring at $x = 0$.

Further, we define f(t) to be of finite average power if

$$0 < \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 \, dt < \infty \quad , \tag{27}$$

in which case

$$g(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(\tau) \, f(t + \tau) \, d\tau \quad . \tag{28}$$

Functions of finite energy or of finite average power therefore satisfy Eq. (2), and if their autocorrelation function satisfies Eqs. (1) and (3), our results hold.† We note here that for signals of finite energy, $S_o(\omega)$ represents their normalized energy density spectrum, whereas for signals of finite average power, $S_o(\omega)$ represents their normalized power density spectrum.

## IV. EXTENSION TO PERIODIC AND DISCRETE TIME FUNCTIONS

In order to extend Theorem 1 (and its corollary) to periodic and/or discrete time functions, we need merely to replace certain integrals in Sec. III with the expressions described below.

For continuous periodic functions, our results hold if we redefine the limits of all previous time integrals to extend over a single period ($T_o$, for example), and if we redefine all integrations with respect to $\omega$ as sums over the discrete set of harmonic frequencies, *viz.*,

$$\int_{-\infty}^{\infty} x(t) \, dt \to \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} x(t) \, dt \quad , \tag{29}$$

and

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} y(\omega) \, d\omega \to \sum_{m=-\infty}^{\infty} y(\omega_m) \quad , \tag{30}$$

where

$$\omega_m = \frac{2\pi m}{T_o} \quad .$$

For these periodic functions, we obtain periodic autocorrelation functions g(t) where

$$g(t) = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} f(\tau) \, f(t + \tau) \, d\tau \quad . \tag{31}$$

In addition, from Eqs. (5), (7), and (30),

$$\sum_{m=-\infty}^{\infty} S_n(\omega_m) = 1 \tag{32}$$

and therefore we must adjust the factor of proportionality by multiplying the right side of Eqs. (23) and (24) by $1/2\pi$.

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† The properties expressed in Eqs. (1) and (3) are most easily stated in terms of g(t) and will be left in that form.

For discrete aperiodic time functions, our results hold if we replace all time integrals with infinite summations over the discrete time variable, and also redefine the limits on all integrations over $\omega$ to extend over the finite range $-(\pi/\Delta t) \leqslant \omega \leqslant (\pi/\Delta t)$, where $\Delta t$ is the uniform increment between adjacent time samples. That is,

$$\int_{-\infty}^{\infty} x(t)\, dt \rightarrow \sum_{m=-\infty}^{\infty} x(t_m) \quad , \tag{33}$$

and

$$\int_{-\infty}^{\infty} y(\omega)\, d\omega \rightarrow \Delta t \int_{-\pi/\Delta t}^{\pi/\Delta t} y(\omega)\, d\omega \quad , \tag{34}$$

where

$$t_m = m\Delta t \quad . \tag{35}$$

For these discrete functions, we obtain discrete autocorrelation functions $g(t_m)$. Furthermore, we find that

$$\int_{-\pi/\Delta t}^{\pi/\Delta t} S_n(\omega)\, d\omega = \frac{2\pi}{\Delta t} \quad , \tag{36}$$

thus, the new factor of proportionality requires that we multiply the right hand side of Eqs. (23) and (24) by $1/\Delta t$.

For discrete functions (of increment $\Delta t$) which are periodic with period $P\Delta t$, we must make the following changes:†

$$\int_{-\infty}^{\infty} x(t)\, dt \rightarrow \sum_{i=0}^{P-1} x(t_i) \tag{37}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} y(\omega)\, d\omega \rightarrow \frac{1}{P} \sum_{m=-P/2}^{P/2} y(\omega_m) \tag{38}$$

where

$$t_i = i\Delta t \quad ,$$

$$\omega_m = \frac{2\pi m}{P\Delta t} \quad .$$

For these discrete periodic functions, we obtain discrete periodic autocorrelation functions $g(t_i)$. Moreover, we have

$$\sum_{m=-P/2}^{P/2} S(\omega_m) = P \quad . \tag{39}$$

Therefore, the new factor of proportionality requires that we multiply the right hand side of Eqs. (23) and (24) by $P/2\pi$.

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† For convenience, we assume P to be an even integer. When P is odd we must alter the limits in Eqs. (38) and (39) slightly.

## V. FINITE SPECTRAL RESOLUTION – A FEASIBLE TIME DOMAIN PROCEDURE

The interesting feature of our iterated autocorrelation function is that one may determine the energy peak (at $\omega = \Theta$, for example) of a signal with operations strictly in the time domain. In practice, one usually has a signal $f(t)$ of finite duration (T) to consider.† As a result, the autocorrelation of the signal $R_o(t)$ will be zero outside the interval $|t| \geq T$. However, the formal procedure for calculating $R_n(t)$ as described by Eq. (5) indicates that $R_n(t)$ will, in general, be nonzero in the interval $|t| \leq 2^n T$. From a practical point of view, this requires an exponentially increasing complexity (in either equipment or computation). At the same time, we have an exponential rate of convergence to our limit function where the exponent is $2^n$ [as may be seen from Eq. (10)]. It is clear that this rate of convergence, although exponential, is nevertheless dependent upon the shape of $S_o(\omega)$. Note, however, that the resolution is (theoretically) perfect, i.e., we are guaranteed to converge exactly on the value $\Theta$.

The price for perfect resolution is, as always, extremely high and one which we are not willing (or able) to pay; namely, that we require an unbounded number of calculations if we insist on passing to the limit $n \to \infty$. Two separate aspects of the complexity of the process grow without bound: the number of iterations n, and the range $|t| \leq 2^n T$ in which $R_n(t)$ must be calculated. If we are willing to sacrifice some resolution, then we may control both of these quantities as follows.

Let $f(t)$ be the signal whose energy peak (at $\omega = \Theta$) we desire, where we assume $f(t)$ to be of finite duration T sec, i.e., $f(t) \equiv 0$ for $t < 0$ and $t > T$. For this function, we make the additional assumptions stated in Eqs. (1) through (3). Define $R_n(t)$ and $S_n(\omega)$ as in Eqs. (4) through (7). We introduce the periodic function $f_p(t)$ of period $T_o \geq 2T$ such that

$$f_p(t + kT_o) = f(t) \quad , \quad 0 \leq t \leq T_o \quad , \quad k = \ldots, -2, -1, 0, 1, 2, \ldots \quad . \tag{40}$$

For this function, we define $S_n^{(P)}(\omega_m)$ and $R_n^{(P)}(t)$, as in Eqs. (4) through (7), with the changes indicated in Eqs. (29) through (32); the superscript P, serves to distinguish these from $S_n(\omega)$ and $R_n(t)$ which refer to $f(t)$.

Below, in Theorem 2, we show that an estimate $\hat{\Theta}$ of the energy peak in $f_p(t)$ is a close approximation to $\Theta$ when $T_o$ is sufficiently large. In addition, we describe a procedure for obtaining this estimate to within a given accuracy after a finite number $(n_o)$ of iterations. Furthermore, since periodic functions have periodic autocorrelation functions, we recognize that $R_n^{(P)}(t)$ is periodic (with period $T_o$) for all n. Consequently, we need carry out our calculations of $R_n^{(P)}(t)$ over only one period. Thus, by agreeing to estimate $\Theta$ within a slight (arbitrarily small) error, we have been able to reduce our system to one which goes through a finite number of iterations over a finite interval $(0, T_o)$ which is fixed with respect to the number of iterations. All this, of course, comes about by allowing an uncertainty of size ($\rho$, let us say) in the estimate of the frequency of maximum energy density.

The relationship between the time limited signal $f(t)$ and its periodic counterpart $f_p(t)$ needs further elaboration. Indeed, we recognize that the autocorrelation functions of these two signals obey an inverse of the sampling theorem. Specifically, since $T_o \geq 2T$, $R_o^{(P)}(t)$ contains all the information about $R_o(t)$; consequently, $S_o^{(P)}(\omega_m)$ must be proportional to $S_o(\omega)$ at $\omega = \omega_m = 2\pi m/T_o$. We may calculate this factor of proportionality from Eqs. (6) and (29), viz.,

† Therefore, the signal will be of finite energy [see Eq. (25)].

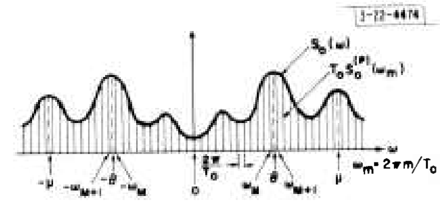
$$\frac{S_o^{(P)}(\omega_m)}{S_o(\omega_m)} = \frac{\frac{1}{T_o}\int_{-T_o/2}^{T_o/2} R_o^{(P)}(t)\, e^{-j\omega_m t}\, dt}{\int_{-\infty}^{\infty} R_o(t)\, e^{-j\omega_m t}\, dt} \quad .$$

The integrand in the numerator disappears in the interval $T < |t| < T_o/2$ and the integrand in the denominator disappears over the interval $|t| > T$. Therefore, since $(T_o/2) \geqslant T$, we find that

$$S_o^{(P)}(\omega_m) = \frac{1}{T_o} S_o(\omega_m) \quad . \tag{41}$$

Recognizing that these two spectra are proportional is essential to one's understanding of Theorem 2. A typical case is shown in Fig. 1, where the continuous spectrum $S_o(\omega)$ is shown as the continuous envelope, and the line spectrum $T_o S_o^{(P)}(\omega_m)$ is shown as vertical lines.

Fig. 1. Comparison between $S_o(\omega)$ and $S_o^{(P)}(\omega_m)$.




Observe that the distance between "samples" of $S_o(\omega)$ is $2\pi/T_o$. We define an integer $M$ such that $\omega_M$ and $\omega_{M+1}$ surround the spectral peak at $\omega = \Theta$ (the same clearly holds true for $-\omega_M$, $-\Theta$ and $-\omega_{M+1}$). Further, we define $\mu$ to be that frequency at which $S_o(\omega)$ has its second greatest local maximum.

In estimating $\Theta$, two separate problems confront us: first, how do we eliminate competing peaks (such as at $\omega = \pm\mu$); and second, how do we converge to the neighborhood of $\Theta$ within the absolute peak itself. The first problem may be solved by insisting that the spacing $2\pi/T_o$ be fine enough to insure that at least one sample in the neighborhood of $\Theta$ (either $\omega_M$ or $\omega_{M+1}$) is greater than the maximum possible sample near $\omega = \mu$. Thus, an important parameter of the spectrum $S_o(\omega)$ may be expressed as a lower bound $\Delta$, for the difference between these two peaks, viz.,

$$S_o(\Theta) - S_o(\mu) \geqslant \Delta \quad . \tag{42}$$

In considering the problem of convergence in the neighborhood of $\Theta$, we recognize that the shape of the spectrum in this region is crucial. Specifically, we require that the spacing $2\pi/T_o$ be small enough to guarantee that some lines in the discrete spectrum fall within the range of the dominant peak (the narrower the peak, the finer must be the spacing). On the other hand, too many spectral lines in the immediate vicinity of $\Theta$ will result in a large number of required iterations, because the largest sample will not differ significantly from its neighbors. We may discuss the width or sharpness of the spectrum in the vicinity of its absolute peak by considering the curvature of $S_o(\omega)$ in terms of the magnitude of its second derivative $d^2S_o(\omega)/d\omega^2$. When the second derivative is large (in magnitude), then the peak is narrow, and vice versa. Thus, we are led to consider upper and lower bounds for this quantity in the neighborhood of $\Theta$, viz.,

$$-A \leqslant \frac{d^2 S_o(\omega)}{d\omega^2} \leqslant -a \quad \text{for } |\omega - \Theta| \leqslant \frac{4\pi}{T_o} \quad . \tag{43}$$

The preceding discussion deals qualitatively with those factors that determine the required number of iterations and the spacing of samples in the discrete spectrum.† In addition to these considerations, the sampling must be fine enough to satisfy the spectral resolution requirements of the user of these results. Stated precisely, we have the following theorem.

**Theorem 2.**

Consider any function f(t) of duration T sec, with a unique absolute energy peak at $\omega = \pm\Theta$, whose energy density spectrum $S_o(\omega)$ satisfies the conditions of **Eqs. (42)** and (43) with respect to the three positive parameters $(a, A, \Delta)$. Then, for a given required frequency resolution $\rho$, there exists a procedure (defined below) which will calculate a number $\hat{\Theta}$ after $n_o$ iterations of the autocorrelation of $f_p(t)$ [see Eq. (40)] over the range $|t| \leqslant T_o$ such that

$$|\hat{\Theta} - \Theta| \leqslant \rho$$

where

$$n_o = \log_2 \frac{1}{2\epsilon_o} \quad , \tag{44}$$

and

$$\epsilon_o = \frac{2\pi^2}{T_o^3} a \quad , \tag{45}$$

and

$$T_o = \max\left(\pi \sqrt{\frac{A + 4a}{2\Delta}}\ , \frac{5\pi}{\rho}\ , 2T\right) \quad . \tag{46}$$

**Proof.**

We begin by describing the procedure by which we determine the estimate $\hat{\Theta}$ from $R_n^{(P)}(t)$. We recognize that if one line (say at $\omega_m = \Theta_p$) of the spectrum of the periodic function $R_n^{(P)}(t)$ is "sufficiently" large‡ compared to all the other lines, then $R_n^{(P)}(t)$ will appear as a "noisy" cosine wave at frequency $\Theta_p$ rad/sec. Our procedure, then, is to count the number of "noisy" zero-crossings of this function for a known time interval; if we are successful in counting only the true zero-crossings of the pure cosine wave, we can then ascertain $\Theta_p$ exactly. We define a noisy zero-crossing counter as follows. Consider a $\delta$-threshold detector with hysteresis defined by the transfer characteristic shown in Fig. 2. Further, define:

Z(t) = number of zeros (or counts) recorded by the noisy zero-crossing counter in t sec

and

Z(0) = 1 .

† The sample spacing may be adjusted by changing the value of $T_o$, i.e., $|\omega_m - \omega_{m-1}| = 2\pi/T_o$.

‡ It is clear that by increasing $T_o$, $\Theta_p$ can be made as close to $\Theta$ as one desires; also, by increasing the number of iterations, $S_n^{(P)}(\Theta_p)$ can be made arbitrarily large compared to all other spectral components (see Theorem 1).

The noisy zero-crossing counter consists of a $\delta$-threshold detector with hysteresis followed by a simple counter which registers a count each time the detector changes state (in either direction) as shown in Fig. 3. It will be shown later that a setting of $\delta = 1/9$ defines a noisy zero-crossing counter which produces an estimate $\hat{\Theta}$ consistent with Theorem 2, that is, $\delta_n \leqslant \delta = 1/9$ for $n \geqslant n_o$ [where $\delta_n$ is defined by Eq. (48) below].

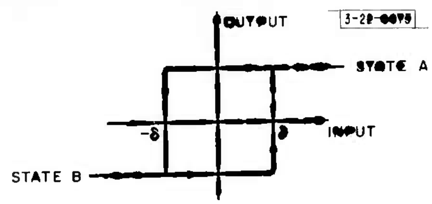



Fig. 2. Transfer characteristic of the $\delta$-threshold detector with hysteresis.

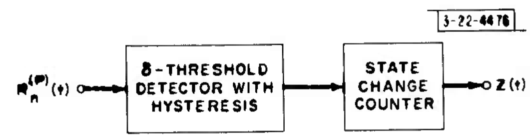



Fig. 3. The noisy zero-crossing counter.

With no loss of generality, we may write

$$R_n^{(P)}(t) = B_n \cos \Theta_p t + b_n(t) \quad . \tag{47}$$

This form exposes the pure cosine wave that we wish to detect, and groups the transform of all the other spectral components into the function $b_n(t)$. Define $\delta_n$ by

$$|b_n(t)| < \delta_n \quad \text{for } |t| \leqslant T_o \quad . \tag{48}$$

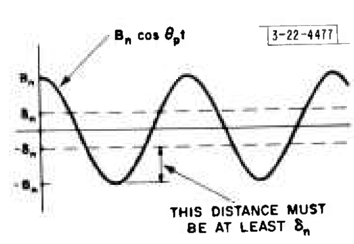
Then, if

$$B_n \geqslant 2\delta_n \quad , \tag{49}$$

we are guaranteed that $Z(t)$ will increase by unity each time $B_n \cos \Theta_p t$ passes through zero, as may be seen in Fig. 4. That is, when $B_n \cos \Theta_p t$ is negative $R_n^{(P)}(t) < \delta_n$ and when $B_n \cos \Theta_p t$ is positive $R_n^{(P)}(t) > -\delta_n$, thus insuring that a single zero-crossing (which is equivalent to a sign change) of $B_n \cos \Theta_p t$ cannot generate more than one count. Furthermore, by insisting that $B_n \geqslant 2\delta_n$, we guarantee that each time $B_n \cos \Theta_p t$ passes through zero, we must get at least one count. Thus, $Z(t) - 1$ will count the number of zero-crossings of $B_n \cos \Theta_p t$. Now, since $T_o$ is a multiple of the period of $B_n \cos \Theta_p t$, we determine $\Theta_p$ from

$$\Theta_p = \frac{Z(T_o) - 1}{T_o} \pi \quad . \tag{50}$$

Fig. 4. Description of the relation between $B_n$ and $\delta_n$.



Generally, if we observe $Z(t)$ for a time $\tau$, where $0 \leqslant \tau \leqslant T_o$, then we may ask how large an error is made in our estimation of $\Theta_p$; in particular, how large is $|\Theta_p - [Z(\tau)/\tau]\,\pi|$. To answer this, we use Euclid's algorithm to express

$$\frac{\tau}{P} = q + \frac{r}{P}$$

where P is the period of $\cos \Theta_p t$ (i.e., $P = 2\pi/\Theta_P$) and q and r are, respectively, the quotient and remainder of $\tau/P$, where, of course, $r < P$. But, $q = [Z(\tau) - 1]/2$ since we increment $Z(t)$ twice for each period P of $\cos \Theta_p t$. Thus,

$$\frac{\tau}{P} = \frac{Z(\tau) - 1}{2} + \frac{r}{P}$$

or, multiplying by $2\pi/\tau$, we obtain

$$\frac{2\pi}{P} = \frac{\pi Z(\tau)}{\tau} + \frac{\pi}{\tau}\left(2\frac{r}{P} - 1\right) \quad .$$

Now, since $\Theta_p = 2\pi/P$, and since $r < P$, we have

$$\left|\Theta_p - \frac{Z(\tau)}{\tau}\pi\right| \leqslant \frac{\pi}{\tau} \quad . \tag{51}$$

Thus, we conclude that the noisy zero-crossing counter determines from $R_n^{(P)}(t)$ the frequency $\Theta_p$ to within the accuracy described by Eq. (51). The error in this determination may be made arbitrarily small by increasing $\tau$.

We now obtain bounds for the magnitude of the difference between spectral components in the neighborhood of $\Theta$. By Taylor's development, we have

$$S_o(\omega) = S_o(\omega') + (\omega - \omega')\frac{dS_o(\omega')}{d\omega} + \frac{(\omega - \omega')^2}{2}\frac{d^2S_o(\sigma)}{d\omega^2} \quad ,$$

where $\sigma$ lies between $\omega$ and $\omega'$. Furthermore, expanding $dS_o(\omega')/d\omega$ we have

$$\frac{dS_o(\omega')}{d\omega} = \frac{dS_o(\Theta)}{d\omega} + (\omega' - \Theta)\frac{d^2S_o(\alpha)}{d\omega^2}$$

where $\alpha$ lies between $\omega'$ and $\Theta$. Combining these two equations, and recognizing that the slope of $S_o(\omega)$ is zero for $\omega = \Theta$, we obtain

$$S_o(\omega) - S_o(\omega') = (\omega - \omega')(\omega' - \Theta)\frac{d^2S_o(\alpha)}{d\omega^2} + \frac{(\omega - \omega')^2}{2}\frac{d^2S_o(\sigma)}{d\omega^2} \quad . \tag{52}$$

For future reference, we now establish a lower bound for this difference in the case $\omega' = \omega_M$, $\omega = \omega_{M-1}$. Recall that $\omega_M - \omega_{M-1} = 2\pi/T_o$ where $\omega_M$ is defined as shown in Fig. 1. Thus, Eq. (52) becomes

$$S_o(\omega_{M-1}) - S_o(\omega_M) = -\frac{2\pi}{T_o}(\omega_M - \Theta)\frac{d^2S_o(\alpha)}{d\omega^2} + \frac{2\pi^2}{T_o^2}\frac{d^2S_o(\sigma)}{d\omega^2} \quad .$$

Since $\sigma$ and $\alpha$ both lie within the range $\Theta \pm 4\pi/T_o$, we may apply Eq. (43) and in obtaining a lower bound for the above expression, we may set $\omega_M = \Theta$, yielding

$$\frac{1}{T_o} \left| S_o(\omega_M) - S_o(\omega_{M-1}) \right| \geq \frac{2a\pi^2}{T_o^3} \quad . \tag{53}$$

Note that a similar equation holds for $\omega_{M+1}$ and $\omega_{M+2}$.

We now consider the conditions necessary to insure that the sample at either $\omega_M$ or $\omega_{M+1}$ exceeds the maximum possible sample at $\omega = \mu$. Recognizing that one of the two samples surrounding $\Theta$ must lie within $\pi/T_o$ of $\Theta$, we apply Eq. (52) [with $\omega' = \Theta$, $\omega = \Theta \pm (\pi/T_o)$] and obtain

$$\left| S_o(\Theta \pm \frac{\pi}{T_o}) - S_o(\Theta) \right| = \left| \frac{\pi^2}{2T_o^2} \frac{d^2 S_o(\sigma)}{d\omega^2} \right| \quad .$$

Application of Eq. (43) gives us

$$\left| S_o(\Theta \pm \frac{\pi}{T_o}) - S_o(\Theta) \right| \leq \frac{A\pi^2}{2T_o^2} \quad .$$

Thus, the minimum value of the maximum sample of $S_o^{(P)}(\omega_m)$ near $\Theta$ is

$$\frac{1}{T_o} \left[ S_o(\Theta) - \frac{A}{2} \frac{\pi^2}{T_o^2} \right]$$

where we have made use of Eq. (41). We choose to require, at this point, that this minimum value should exceed $S_o(\mu)/T_o$ by at least the quantity† $\epsilon_o \equiv 2a\,(\pi^2/T_o^3)$. Thus,

$$\frac{1}{T_o} \left[ S_o(\Theta) - \frac{A}{2} \frac{\pi^2}{T_o^2} \right] \geq \frac{S_o(\mu)}{T_o} + 2a \frac{\pi^2}{T_o^3} \quad . \tag{54}$$

Applying Eq. (42), we obtain, as our first condition on $T_o$ [see Eq. (46)],

$$T_o \geq \pi \sqrt{\frac{A+4a}{2\Delta}} \quad . \tag{55}$$

We have eliminated the problem of competing peaks.

We may now discuss convergence of $R_n^{(P)}(t)$, in the neighborhood of $\Theta$. Theorem 1 states‡ that the iterated normalized autocorrelation $R_n^{(P)}(t)$ of $f_p(t)$ converges to $\cos \Theta_p t$ where $S_o^{(P)}(\Theta_p) > S_o^{(P)}(\omega_m)$ for all $\omega_m \neq \pm\Theta_p$. However, as stated in the corollary to Theorem 1, if there are a number of equal absolute maxima, then $R_n^{(P)}(t)$ will converge to a sum of cosines at the various frequencies of the equal peaks. We have postulated that $S_o(\omega)$ has a unique peak at $\omega = \Theta$. Nevertheless, $S_o^{(P)}(\omega_m)$ may have two equal peaks at $\omega_m = \omega_M, \omega_{M+1}$ as shown in Fig. 5. On the other hand, $S_o^{(P)}(\omega_M)$ and $S_o^{(P)}(\omega_{M+1})$ may be arbitrarily close in magnitude, depending upon $S_o(\omega)$ and the frequency sampling rate $2\pi/T_o$. In order to handle this annoying circumstance, we express $R_n^{(P)}(t)$ (again, with no loss of generality) as

$$R_n^{(P)}(t) = C_n \left[ \beta \cos \frac{2\pi t}{T_o} M + (1-\beta) \cos \frac{2\pi t}{T_o} (M+1) \right] + c_n(t) \quad . \tag{56}$$

---

† Note that $\epsilon_o$ is just the lower bound in Eq. (53).

‡ We use this theorem with its extension to periodic functions; see Sec. IV.

This form for $R_n^{(P)}(t)$ exposes the interaction of the two largest components of $S_n^{(P)}(\omega_m)$ and groups the transform of all the other components into a single function $c_n(t)$. The variable $\beta$ allows us

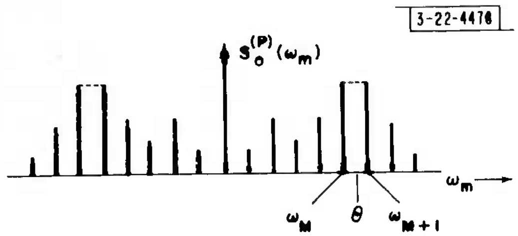



Fig. 5. Two equal peaks in $S_a^{(P)}(\omega_m)$.

to investigate the effect of the relative magnitudes of these two components, where, of course, $0 \leqslant \beta \leqslant 1$. It is clear that $c_n(t)$ is a sum of cosines, each with zero phase;† therefore,

$$c_n(0) \geqslant |c_n(t)| \quad . \tag{57}$$

Furthermore, since $R_n^{(P)}(0) = 1$, we note from Eq. (56) that

$$C_n = 1 - c_n(0) \quad . \tag{58}$$

We concentrate our attention on the function

$$e(\beta, t) = \beta \cos \frac{2\pi t}{T_o} M + (1-\beta) \cos \frac{2\pi t}{T_o} (M+1) \tag{59}$$

and investigate the behavior of its maxima and minima in order to determine the effect of passing $R_n^{(P)}(t)$, as expressed in Eq. (56), through the noisy zero-crossing counter.

Define $S_k$ to be that instant when the time derivative of $e(\beta, t)$ has its $k^{th}$ zero, i.e.,

$$\frac{de(\beta, S_k)}{dt} = 0 \quad .$$

**Lemma 1.**‡

We make use of Lemma 1 as follows. For all $0 \leqslant \beta \leqslant 1$, and for all k and $\tau$ such that§

$$0 \leqslant S_k \leqslant u_k \leqslant \tau \leqslant T_o/2 \quad ,$$

then

$$|e(\tfrac{1}{2}, \tau)| \leqslant |e(\beta, S_k)| \quad . \tag{60}$$

Application of this lemma will be made when we consider Eq. (56), namely,

$$R_n^{(P)}(t) = C_n e(\beta, t) + c_n(t)$$

and compare it to Eq. (47),

$$R_n^{(P)}(t) = B_n \cos \Theta_p t + b_n(t) \quad .$$

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† Recall that $S_n(\omega)$ is always real and even.

‡ See the Appendix for proof of Lemma 1.

§ See the proof of Lemma 1 for a definition of $u_k$.

Whereas $e(\beta, t)$ is not a pure cosine, it does have the same number of relative peaks (maxima and minima) as some cosine of frequency ($\omega'$, for example) where

$$\omega_M \leqslant \omega' \leqslant \omega_{M+1} \quad . \tag{61}$$

The same can be said about the number of zeros of $e(\beta, t)$. We may now write an equation, similar to Eq. (49), which places a condition on the magnitude of the relative peaks of $e(\beta, t)$ over a range of t. Specifically, if we expect the noisy zero-crossing counter to calculate the frequency $\omega'$ when $R_n^{(P)}(t)$ [as expressed in Eq. (56)] is present, then we require [see the proof of Eq. (49)], that

$$C_n \left| e(\beta, S_k) \right| \geqslant 2 \left| c_n(t) \right| \quad .$$

By Lemma 1, therefore, we require that

$$C_n \left| e(\tfrac{1}{2}, \tau) \right| \geqslant 2 \left| c_n(t) \right|$$

for $0 \leqslant S_k \leqslant u_k \leqslant \tau \leqslant T_o/2$. By Eqs. (57) and (58) we obtain, as an equivalent condition,

$$\left| e(\tfrac{1}{2}, \tau) \right| \geqslant 2 \frac{c_n(0)}{1 - c_n(0)} \quad , \quad 0 \leqslant \tau \leqslant \frac{T_o}{2} \quad . \tag{62}$$

Thus, we propose to choose some $\tau$ in the range† $0 \leqslant \tau \leqslant (T_o/2)$ and agree to observe Z(t) only up to $t = \tau$. In limiting our interval of observation to $\tau$ instead of $T_o$ we incur a slightly greater ambiguity in the estimation of $\omega'$, namely [see Eq. (51)],

$$\left| \omega' - \frac{Z(\tau)}{\tau} \pi \right| \leqslant \frac{\pi}{\tau} \quad . \tag{63}$$

We define $\hat{\Theta}$ to be the output of our noisy zero-crossing counter at time $\tau$, viz.,

$$\hat{\Theta} = \frac{Z(\tau)}{\tau} \pi \quad . \tag{64}$$

Furthermore, since $\omega'$ is bounded between $\omega_M$ and $\omega_{M+1}$ [see Eq. (61)] and since $\Theta$ is in the same interval (of width $\omega_{M+1} - \omega_M = 2\pi/T_o$), we conclude [with the application of Eqs. (63) and (64)] that the maximum error in the estimate ($\hat{\Theta}$) of $\Theta$ is

$$\left| \hat{\Theta} - \Theta \right| \leqslant \pi \left( \frac{2}{T_o} + \frac{1}{\tau} \right) \quad . \tag{65}$$

Consequently, we would like to make $\tau$ and $T_o$ as large as possible; however, note that at the times of interest (namely at the peaks), $|e(1/2, \tau)|$ behaves essentially as $\cos(\pi\tau/T_o)$. Thus, as $\tau$ is increased, the condition expressed in Eq. (62) becomes more difficult to satisfy, therefore $n_o$ must increase (see below). This represents a trade among $n_o$, $T_o$, and $\tau$. To be specific, we choose a reasonable value for $\tau$, namely, $\tau = T_o/3$. Then, from Eq. (59), $|e(1/2, T_o/3)| \geqslant 1/4$ and we obtain the following‡ from Eqs. (62) and (65),

$$\frac{1 - c_n(0)}{c_n(0)} \geqslant 8 \tag{66}$$

† Here we are neglecting the restriction that $u_k \leqslant \tau$; in so doing, we make a slight error. However, this error is insignificant for large M ($M \geqslant 10$, typically).

‡ Equation (67) therefore requires $5\pi/T_o \leqslant \rho$. See Eq. (46).

and

$$|\hat{\theta} - \theta| \leq \frac{5\pi}{T_o} \quad . \tag{67}$$

Note that Eq. (66) specifies an upper bound for $c_n(0)$; we may therefore use this as the setting for the threshold in our noisy zero-crossing counter. That is, $\delta = 1/9$.

We now consider the behavior of $c_n(0)$. Specifically, $c_n(t)$ is the transform of all spectral lines except the two pairs at $\omega = \pm\omega_M, \pm_{M+1}$. Therefore, $c_n(0)$ is merely the sum of the magnitudes of all such lines, viz.,

$$c_n(0) = \sum_{m=-\infty}^{\infty} S_n^{(P)}(\omega_m) - S_n^{(P)}(\omega_M) - S_n^{(P)}(-\omega_M) - S_n^{(P)}(\omega_{M+1}) - S_n^{(P)}(-\omega_{M+1}) \quad . \tag{68}$$

We now consider the worst† case, namely,

$$S_n^{(P)}(\omega_M) = S_n^{(P)}(\omega_{M+1}) \quad .$$

Since $S_n^{(P)}(\omega_m) = S_n^{(P)}(\omega_{-m})$, we observe‡ that the coefficient $C_n$ [as defined by Eq. (56)] is

$$C_n = 4S_n^{(P)}(\omega_M) \quad .$$

Furthermore, since $\sum_{m=-\infty}^{\infty} S_n^{(P)}(\omega_m) = 1$ [see Eq. (32)], we may rewrite Eq. (68) as

$$c_n(0) = 1 - 4S_n^{(P)}(\omega_M) = 1 - C_n \quad . \tag{69}$$

The behavior of $S_n^{(P)}(\omega_M)$ [recall that $S_o^{(P)}(\omega_M) > S_o^{(P)}(\omega_m)$ for all $m \neq \pm M, \pm(M+1)$] may be determined by considering the extension of Eq. (19) to periodic functions, viz.,

$$S_n^{(P)}(\omega_M) = \frac{[S_o^{(P)}(\omega_M)]^{2^n}}{\sum_{m=-\infty}^{\infty} [S_o^{(P)}(\omega_m)]^{2^n}} \quad .$$

For convenience, we temporarily adopt the notation,

$$h_m = S_o^{(P)}(\omega_m) \quad .$$

Thus, Eq. (69) becomes

$$c_n(0) = 1 - \frac{4h_M^{2^n}}{\sum_{m=-\infty}^{\infty} h_m^{2^n}} \quad . \tag{70}$$

We adopt the further notation,

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† That is, the case which requires the largest $n_o$. This corresponds to $\beta = 1/2$.

‡ Note that the transform of any corresponding pair of spectral lines, say $S_n^{(P)}(\omega_m)$ and $S_n^{(P)}(\omega_{-m})$, is merely $2S_n^{(P)}(\omega_m) \cos \omega_m t$, since all components of $S_n^{(P)}(\omega_m)$ are real and positive.

$$\sum_m{}' = \sum_{m \neq \pm M, \pm(M+1)} .$$

Equation (70) then may be written as

$$c_n(0) = 1 - \frac{4h_M^{2^n}}{4h_M^{2^n} + \sum_m{}' h_m^{2^n}} . \tag{71}$$

We recognize that $c_n(0)$ (and therefore, $n_o$) is maximized when, for fixed $h_M = S_o^{(P)}(\omega_M)$, the sum $\sum_m{}' h_m^{2^n}$ is maximized. In order to find the maximum of this sum, we find it convenient to state and solve an equivalent problem, as stated in Lemma 2 below. In this lemma we recognize that the quantities $\Sigma b_k$, $a_1$ and $N$ correspond respectively to $\Sigma' h_m$, $1 - 4h_M$ and $2^n$. The value of $a_2$ will become apparent shortly.

**Lemma 2.**†

Given $\{b_k\}$ $k = 1, 2, 3, \ldots,$ such that

$$\sum_{k=1}^{\infty} b_k = a_1 ,$$

and

$$0 \leq b_k \leq a_2 < a_1 .$$

Then the set $\{b_k^*\}$ which maximizes

$$\sigma = \sum_{k=1}^{\infty} b_k^N$$

is

$$b_k^* = \begin{cases} a_2 & k = 1, 2, \ldots, K-1 \\ a_1 - (K-1)\, a_2 & k = K \\ 0 & k > K \end{cases} \tag{72}$$

where

$$K = \left[\frac{a_1}{a_2}\right] + 1 .$$

and $[x]$ = the maximum integer contained in x.

Maximizing $c_n(0)$ corresponds to maximizing $n_o$ [i.e., it makes the inequality in Eq. (66) most difficult to satisfy]. This "worst" condition as described in Lemma 2 may be combined with Eq. (71) to give

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† The proof of Lemma 2 will be found in the Appendix.

$$c_n(0) \leq 1 - \frac{4h_M^{2^n}}{4h_M^{2^n} + (K-1)\, a_2^{2^n} + \{a_1 - (K-1)\, a_2\}^{2^n}} \quad . \tag{73}$$

We recognize that

$$a_1 = 1 - 4h_M$$

$$K - 1 = \left[\frac{1 - 4h_M}{a_2}\right] \quad .$$

As for $a_2$, which represents the maximum value of $S_o^{(P)}(\omega_m)$ for $m \neq \pm M,\ \pm(M+1)$, we refer back to Eq. (53) and to the discussion preceeding Eq. (54) to obtain

$$a_2 = h_M - \frac{2a\pi^2}{T_o^3} \quad . \tag{74}$$

A portion of the denominator of Eq. (73) may be bounded above as follows:

$$(K-1)\, a_2^{2^n} + \{a_1 - (K-1)\, a_2\}^{2^n} = a_2^{2^n} \left\{\left[\frac{1 - 4h_M}{a_2}\right] + \left(\frac{1-4h_M}{a_2} - \left[\frac{1-4h_M}{a_2}\right]\right)^{2^n}\right\}$$

$$\leq a_2^{2^n} \left(\frac{1-4h_M}{a_2}\right) \quad .$$

Thus, Eq. (73) becomes

$$c_n(0) \leq 1 - \frac{1}{1 + \left(\frac{1-4h_M}{4a_2}\right)\left(\frac{a_2}{h_M}\right)^{2^n}} \quad . \tag{75}$$

Recalling the definition

$$\epsilon_o \equiv \frac{2a\pi^2}{T_o^3} \quad , \tag{76}$$

we substitute Eq. (74) in Eq. (75) to obtain

$$c_n(0) \leq 1 - \frac{1}{1 + \left(\frac{1-4h_M}{4h_M - 4\epsilon_o}\right)\left(\frac{h_M - \epsilon_o}{h_M}\right)^{2^n}} \quad . \tag{77}$$

We now consider that value of $h_M$ ($H_M$, let us say) which maximizes

$$E = \frac{1-4h_M}{4h_M - 4\epsilon_o} \left(\frac{h_M - \epsilon_o}{h_M}\right)^{2^n}$$

$$= \left(\frac{1}{4h_M} - 1\right)\left(1 - \frac{\epsilon_o}{h_M}\right)^{2^n - 1} \quad . \tag{78}$$

Differentiating, we get

$$\frac{dE}{dh_M} = -\frac{1}{4h_M^2}\left(1-\frac{\epsilon_o}{h_M}\right)^{2^n-1} + (2^n-1)\left(\frac{1}{4h_M}-1\right)\left(1-\frac{\epsilon_o}{h_M}\right)^{2^n-2}\left(\frac{\epsilon_o}{h_M^2}\right) .$$

Setting this derivative equal to zero, we find that

$$H_M = \frac{2^n\epsilon_o}{1+4\epsilon_o(2^n-1)} .$$

We note that $\epsilon_o \leqslant H_M \leqslant 1/4$ as of course it must. Substituting the value $h_M = H_M$ into Eq. (78) yields

$$E \leqslant \left(\frac{1+4\epsilon_o(2^n-1)}{4\epsilon_o 2^n} - 1\right)\left(1-\frac{1+4\epsilon_o(2^n-1)}{2^n}\right)^{2^n-1}$$

$$= \frac{1}{4}\left(\frac{1-4\epsilon_o}{2^n\epsilon_o}\right)(1-2^{-n}-4\epsilon_o+4\epsilon_o 2^{-n})^{2^n-1}$$

$$\leqslant \frac{2^{-n}}{4\epsilon_o}(1-4\epsilon_o)^{2^n} .$$

Thus, Eq. (77) becomes

$$c_n(0) \leqslant 1 - \frac{1}{1+\frac{2^{-n}}{4\epsilon_o}(1-4\epsilon_o)^{2^n}} . \tag{79}$$

Returning to the condition expressed in Eq. (66), and using Eq. (79) we obtain as our condition, after some algebra,

$$(1-4\epsilon_o)^{2^n} \leqslant \frac{\epsilon_o 2^n}{2} , \tag{80}$$

where $\epsilon_o$ is defined in Eq. (76). Rewriting Eq. (80) in terms of logarithms, we obtain

$$n + 2^n \log_2 \frac{1}{1-4\epsilon_o} \geqslant \log_2 \frac{2}{\epsilon_o} . \tag{81}$$

However, due to the convexity of the function $\log 1/(1-x)$ in the interval $0 \leqslant x < 1$, we may bound $\log 1/(1-x)$ from below by its tangent at $x = 0$, viz.,

$$\log_2 \frac{1}{1-x} \geqslant \left[\left(\frac{d}{dx}\log_2\frac{1}{1-x}\right)_{x=0}\right]\cdot x$$

or

$$\log_2 \frac{1}{1-x} \geqslant x \log_2 e .$$

Applying this inequality to Eq. (81) results in a slightly larger value of $n_o$. Therefore, we obtain

$$n + 2^{n+2}\epsilon_o \log_2 e \geqslant \log_2 \frac{2}{\epsilon_o} . \tag{82}$$

We now define

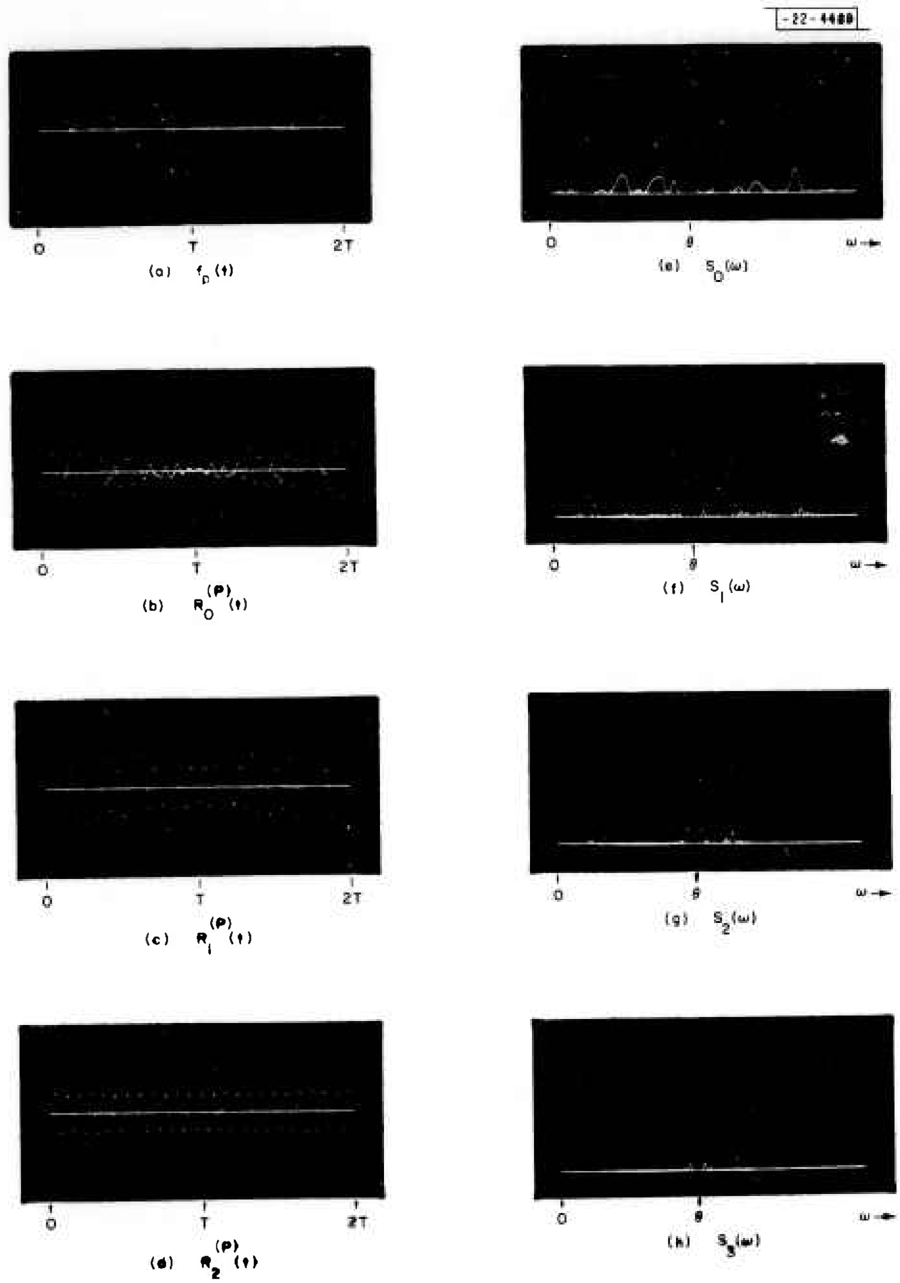


**Fig. 6.** Power spectrum $S_n(\omega)$ and one period of the iterated autocorrelation function $R_n^{(P)}(t)$ as a function of the number of iterations $n$.

$$y = \log_2 \frac{2}{\epsilon_o}$$

and so

$$\epsilon_o = 2^{1-y} \quad .$$

Eq. (82) then becomes

$$n + 2^{n+3-y} \log_2 e \geqslant y \quad . \tag{83}$$

The minimum value of n ($n_o$, for instance) which satisfies Eq. (83) is clearly $n_o = y - 2$, so

$$n_o = \log_2 \frac{2}{\epsilon_o} - 2 = \log_2 \frac{1}{2\epsilon_o} \quad . \tag{84}$$

Thus, $n_o$ represents the number of iterations of the normalized autocorrelation of $f_p(t)$ which we need in order to obtain a number $\hat{\theta}$ satisfying Eq. (67). This completes the proof of Theorem 2.

## VI. EXPERIMENTAL RESULTS

The procedure described in Sec. V for determining the energy peak of a finite duration signal to a finite resolution was simulated on a digital computer.† Results of the experimentation are shown in Fig. 6.

Specifically, the signal f(t) chosen was a small segment (T = 0.0256 sec) of human speech sampled at a 10 kc rate. Figure 6 shows only one period, the interval $0 \leqslant t \leqslant T_o$, of $f_p(t)$ and $R_n^{(P)}(t)$ for n = 0, 1, 2, as well as $S_n(\omega)$ for n = 0, 1, 2, 3. $T_o$ was chosen equal to 2T for this experiment. Note that $S_n(\omega)$ is shown only as a visual aid; its calculation was clearly not necessary for the generation of $R_n^{(P)}(t)$. We chose to show $S_n(\omega)$ rather than $S_n^{(P)}(\omega_m)$ for convenience of programming; as a result, we observe the $\{[\sin a(\omega - \omega_m)]/a(\omega - \omega_m)\}^2$ envelope quite distinctly in $S_n(\omega)$.

One clearly observes the rapid convergence of both $R_n^{(P)}(t)$ and $S_n(\omega)$ to the frequency of maximum energy density.

## VII. APPLICATIONS AND CONCLUSIONS

Theorem 1 expresses the fundamental result that the limit of the iterated normalized autocorrelation function of a signal is a cosine wave at a frequency ($\Theta$) corresponding to the maximum energy peak of the signal's spectrum. However, two aspects of the procedure by which one arrives at this limit require unbounded complexity: first, the interval over which the $n^{th}$ iteration must be calculated (assuming the signal to be of finite duration T) is $|t| \leqslant 2^n T$; second, the number of iterations grows without bound. These two difficulties require unlimited equipment and time, respectively. Clearly, the reason behind these infinite operations is that we are asking for an absolutely perfect determination of $\Theta$. Naturally, we are willing to accept some error in this determination in any practical situation. Taking advantage of this fact, we are able to establish Theorem 2 in which we offer a procedure for estimating $\Theta$ (to within an arbitrarily small, but finite, error) which requires a finite number of calculations over a fixed time interval. Thus, by accepting an error in the determination of $\Theta$, we have been able to eliminate both undesirable aspects of the original procedure.

† A program for simulating the iterated autocorrelation written by the author, and a spectrum analysis program written by C. Rader, were run on Lincoln Laboratory's TX-2 high-speed digital computer (see Ref. 3).

The procedure for obtaining the estimate ($\hat{\Theta}$) of $\Theta$ may be mechanized as follows (see Fig. 7). The signal f(t) would be stored on a tapped delay line (of $T_o$ seconds). At time $T_o$, the impulse response of a linear filter† would be set equal to the values of the taps and the output of the delay line would then feed into the filter. The output of the filter during the interval $2T_o \leqslant t \leqslant 3T_o$ (now equal to the autocorrelation of the signal) would then be fed back into the delay line. The procedure is repeated $n_o$ times, after which the output of the filter is sent through the noisy zero-crossing counter, which provides the estimate $\hat{\Theta}$.

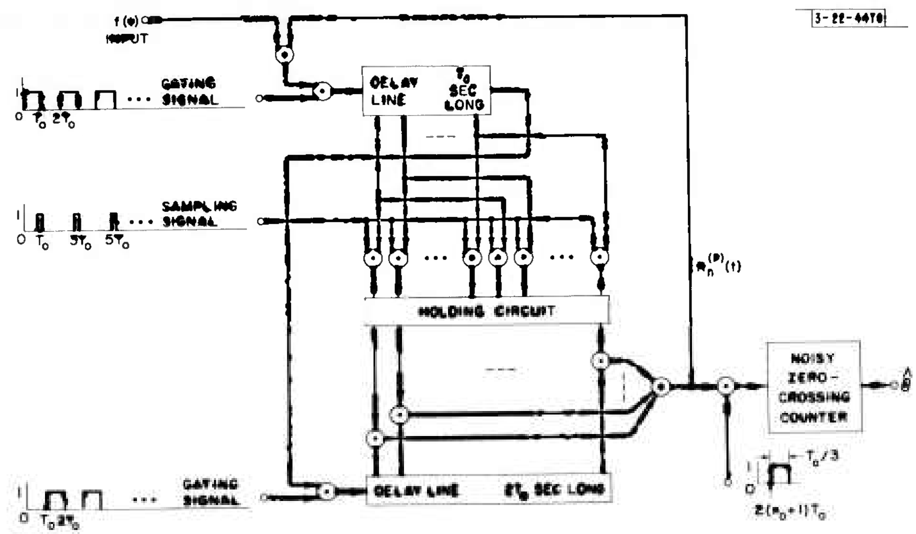


Fig. 7. Implementation of the procedure for detecting the frequency of maximum energy density.

The applications of these results appear to be numerous. The detection of the energy peak of an arbitrary signal (for example, the energy peak corresponding to a time series) is often of great interest, and may be detected as above. Furthermore, this method may be used for selecting that one out of N possible signals transmitted over a communication link; in this application it is necessary, of course, that the location of the absolute peaks in the spectra of the N signals be distinct, one from the other, such as is the case in Frequency Shift Keying. Another application may be found in locating the peak frequency in the return signal from a Doppler radar. These are but a few of the possible applications of the procedure described.

The main conclusion to be drawn from this study is that the iterated autocorrelation procedure represents a new method for detecting the frequency of maximum energy density of a signal. Some areas of application have been suggested above briefly; but, more careful consideration and analysis must be undertaken before one can determine the advantages or disadvantages of this system compared to any other. In fact, one of the main purposes of presenting this material is to stimulate thinking about the possible applications and merits of this new detection approach.

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† This filter is $2T_o$ sec long, with its impulse response h(t) adjusted such that $h(t + T_o) = h(t)$, thus representing the periodic version of the signal. In Fig. 7, this linear filter is represented as a combination of a holding circuit and a second tapped delay line, the outputs of which are multiplied and summed to form the output of the linear filter.

## APPENDIX
## PROOFS OF LEMMAS

**Lemma 1.**

We may express $e(\beta, t)$ as the real part of the sum of two vectors in the complex plane, viz.,

$$e(\beta, t) = \text{Re}\left\{\beta \exp[j(2\pi t/T_o)M] + (1-\beta)\exp[j(2\pi t/T_o)(M+1)]\right\}$$

$$= \text{Re}\,[A(\beta, t)\, e^{j\gamma(\beta, t)}] \quad ,$$

where $A(\beta, t)$ is the magnitude of this vector sum, and $\gamma(\beta, t)$ is its phase angle. We may express $A(\beta, t)$ and $\gamma(\beta, t)$, after some trigonometric simplifications, as

$$A(\beta, t) = \left[\beta^2 + (1-\beta)^2 + 2\beta(1-\beta)\cos\frac{2\pi t}{T_o}\right]^{1/2} \tag{A-1}$$

and

$$\gamma(\beta, t) = \frac{2\pi t}{T_o} M + \tan^{-1}\frac{(1-\beta)\sin\frac{2\pi t}{T_o}}{\beta + (1-\beta)\cos\frac{2\pi t}{T_o}} \quad . \tag{A-2}$$

Forming $dA(\beta, t)/d\beta = 0$, we easily show that

$$A(\tfrac{1}{2}, t) \leqslant A(\beta, t) \quad . \tag{A-3}$$

Furthermore, we observe that for $0 \leqslant t \leqslant T_o/2$,

$$\frac{dA(\beta, t)}{dt} < 0 \quad ,$$

from which we conclude that

$$A(\beta, t_1) \geqslant A(\beta, t_2) \qquad \text{for } t_1 \leqslant t_2 \leqslant \frac{T_o}{2} \quad . \tag{A-4}$$

Let $u_k$ be that instant when $\gamma(\beta, u_k) = (k-1)\pi$. Now, for $0 \leqslant t \leqslant (T_o/2)$, we observe that $d\gamma(\beta, t)/dt$ is always positive; furthermore, since $A(\beta, t)$ is a decreasing function of $t$ in this interval [see Eq. (A-4)], we note that $de(\beta, t)/dt$ passes through zero each time the vector $A(\beta, t)\, e^{j\gamma(\beta, t)}$ passes through either the second or fourth quadrants of the complex plane. We may then conclude that the instants $u_k$ and $S_k$ must alternate, viz.,

$$\ldots \leqslant u_{k-1} \leqslant S_k \leqslant u_k \leqslant S_{k+1} \leqslant \ldots \quad .$$

In addition,

$$|e(\beta, S_k)| \geqslant |e(\beta, u_k)| = A(\beta, u_k) \quad , \qquad \text{A-5)}$$

which follows from the definition of $S_k$ and $u_k$ and the fact that the relative maxima and minima of $e(\beta, t)$ occur when the vector $A(\beta, t)\, e^{j\gamma(\beta, t)}$ passes through the second and fourth quadrants. It is also clear that

$$|e(\beta, t)| = |A(\beta, t)\cos\gamma(\beta, t)| \leqslant A(\beta, t) \quad . \tag{A-6}$$

From Eqs. (A-3) through (A-6) we may write

$$|e(\beta, S_k)| \geqslant |e(\beta, u_k)| = A(\beta, u_k) \geqslant A(\tfrac{1}{2}, u_k) \geqslant A(\tfrac{1}{2}, \tau) \geqslant |A(\tfrac{1}{2}\tau) \cos\gamma(\tfrac{1}{2}, \tau)| = |e(\tfrac{1}{2}, \tau)| \quad ,$$

where

$$0 \leqslant S_k \leqslant u_k \leqslant \tau \leqslant \frac{T_o}{2} \quad .$$

This completes the proof of Lemma 1.

**Lemma 2.**

Consider a sequence of sets $\{b_k^{(i)}\}$ where we define

$$\{b_k^{(0)}\} = \{b_k\} \quad .$$

Corresponding to each such sequence, we define

$$\sigma_i = \sum_{k=1}^{\infty} [b_k^{(i)}]^N$$

where, obviously, $\sigma_o = \sigma$. Since we have freedom in labelling the subscript $k$, we choose to arrange the $b_k^{(0)}$ in a nonincreasing sequence; that is,

$$b_k^{(0)} \geqslant b_{k+1}^{(0)} \quad .$$

The iteration on $i$ may be described essentially as follows. We concentrate on two special values of $k$, say $k_1(i)$ and $k_2(i)$, where

$k_1(i)$ = the smallest value of $k$ for which $b_k^{(i)} < a_2$ ,

$k_2(i)$ = the smallest value of $k$ greater than or equal to $K$, for which $b_k^{(i)} > 0$ .

Corresponding to these values, we define

$$D_i = b_{k_1(i)}^{(i)}$$

$$d_i = b_{k_2(i)}^{(i)} \quad .$$

Then, the set $\{b_k^{(i+1)}\}$ is defined in terms of $\{b_k^{(i)}\}$ by

$$b_k^{(i+1)} = \begin{cases} D_i + \alpha_i d_i & \text{for } k = k_1(i) \\ d_i(1-\alpha_i) & \text{for } k = k_2(i) \\ b_k^{(i)} & \text{otherwise} \end{cases} \qquad \text{(A-7)}$$

where

$$\alpha_i = \min\left(1, \frac{a_2 - D_i}{d_i}\right) \quad .$$

Thus, at each stage, we transfer a fraction $(\alpha_i)$ of $d_i$ to $D_i$ being careful not to allow any $b_k^{(i+1)}$ to exceed $a_2$. Now, it is clear, from the definition of K and Eq. (A-7), that $k_1(i) \leqslant K$ and $k_2(i) \geqslant K$. For those i where $k_1(i)$ remains constant, $D_i$ is an increasing function, and for those i where $k_2(i)$ remains constant, $d_i$ is a decreasing function. Also, since $b_k^{(0)} \geqslant b_{k+1}^{(0)}$ we have

$$D_i \geqslant d_i \quad \text{for all } i \quad .$$

Furthermore, at each step, we conserve the sum

$$\sum_{k=1}^{\infty} b_k^{(i)} = a_1 \quad .$$

The iterative procedure continues until we reach that value of i ($i_o$, let us say) for which

$$b_k^{(i_o)} = a_2 \quad k = 1, 2, \ldots, K-1 \quad . \tag{A-8}$$

We now show that $\sigma_{i+1} \geqslant \sigma_i$ for all $i < i_o$. As a result of Eq. (A-7) we may write

$$\sigma_{i+1} = \sigma_i - D_i^N - d_i^N + (D_i + \alpha_i d_i)^N + [(1-\alpha_i)\, d_i]^N \quad .$$

Using the binomial expansion, we get

$$\sigma_{i+1} - \sigma_i = \sum_{j=0}^{N} \binom{N}{j} [D_i^j(\alpha_i d_i)^{N-j} + d_i^j(-\alpha_i d_i)^{N-j}] - D_i^N - d_i^N$$

$$= \sum_{j=0}^{N-1} \binom{N}{j} (\alpha_i d_i)^{N-j} [D_i^j + (-1)^{N-j} d_i^j] \quad .$$

But, since $D_i \geqslant d_i$, we have, for $i < i_o$,

$$\sigma_{i+1} - \sigma_i \geqslant 0 \quad .$$

We now consider the remainder

$$\sum_{k=K}^{\infty} b_k^{(i_o)} = a_1 - (K-1)\, a_2 \equiv \gamma \quad .$$

But,

$$\sum_{k=K}^{\infty} \left[b_k^{(i_o)}\right]^N \leqslant \left[\sum_{k=K}^{\infty} b_k^{(i_o)}\right]^N = \gamma^N$$

and this upper bound may be achieved when

$$b_k^{(i_o)} = \begin{cases} \gamma & k = K \\ 0 & k > K \end{cases} \quad . \tag{A-9}$$

We observe that

$$\{b_k^{(i_o)}\} = \{b_k^*\} \quad ,$$

where $\{b_k^{(i_o)}\}$ is defined by Eqs. (A-8) and (A-9). Thus, we have shown that starting from any $\{b_k^{(0)}\}$, the sequence $\{b_k^{(i)}\}$ may be defined in a way such that

$$\lim_{i \to i_o} \{b_k^{(i)}\} = \{b_k^*\} \quad ,$$

and

$$\sigma_i \leqslant \sigma_{i+1} \qquad i < i_o \quad .$$

This completes the proof of Lemma 2.

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